Fure akslantirishi yordamida matematik fizika masalalarini yechish

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Butun sonlar o'qida aniqlangan va absolyut integrallanuvchi $f(x)$ funksiyaning Fure integrali [1]

$$f(x)=\int_0^{\infty}\left[a(\lambda)cos\lambda x+b(\lambda)sin\lambda x\right]d\lambda,$$

(1)bu yerda

$$a(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)cos\lambda tdt \quad , \quad b(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)sin\lambda tdt.$$

(2) Eyler formulasi $e^{i\varphi}=cos\varphi+isin\varphi$ dan ,

$$cos\lambda x=\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}, \quad sin\lambda x=\frac{e^{i\lambda x}-e^{-i\lambda x}}{2i},$$

ga teng.

Bularni (1) ga qoysak,

$$f(x)=\int_0^{\infty}\left[a(\lambda)\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}+b(\lambda)\frac{e^{i\lambda x}-e^{-i\lambda x}}{2i}\right]d\lambda=$$

$$=\frac{1}{2}\int_0^{\infty}\left[\left(a(\lambda)-ib(\lambda)\right)e^{i\lambda x}+\left(a(\lambda)+ib(\lambda)\right)e^{-i\lambda x}\right]d\lambda .$$

(3) $c(\lambda)=\pi\left(a(\lambda)-ib(\lambda)\right),$

belgilash kiritib , (2) formulalardan

$$c(\lambda)=\int_{-\infty}^{\infty}f(t)\left(\cos\lambda t-i\sin\lambda t\right)d\lambda=\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt,$$

$$c(\lambda)=\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt ,$$

(4) $c(-\lambda)=\int_{-\infty}^{\infty}f(t)e^{i\lambda t}dt,$

(5) (4) va (5) ni (3) ga qoysak,

$$f(x)=\frac{1}{2\pi}\int_0^{\infty}\left[c(\lambda)e^{i\lambda x}+c(-\lambda)e^{-i\lambda x}\right]d\lambda=\frac{1}{2\pi}\int_0^{\infty}c(\lambda)e^{i\lambda x}d\lambda+$$

$$+\frac{1}{2\pi}\int_0^{\infty}c(-\lambda)e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\int_0^{\infty}c(\lambda)e^{i\lambda x}d\lambda+\frac{1}{2\pi}\int_0^{-\infty}c(\lambda)e^{i\lambda x}d(-\lambda)=$$

$$=\frac{1}{2\pi}\int_0^{\infty}c(\lambda)e^{i\lambda x}d\lambda+\frac{1}{2\pi}\int_{-\infty}^{0}c(\lambda)e^{i\lambda x}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}c(\lambda)e^{i\lambda x}d\lambda\ .$$

Shunday qilip,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}c(\lambda)e^{i\lambda x}d\lambda,$$

(6) bunda

$$c(\lambda)=\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt.$$

(6) formula *Fure integralining kompleks formasi* dep ataladi. Fure integralining kompleks ko'rinishi ikki karrali integral ko'rinishida quyidagicha yoziladi [2]:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}f(t)e^{i\lambda(x-t)}dt.$$

(7)Fure akslantirishi:

$f(x)$ funksiya butun sonlar o'qida aniqlangan, $(-\infty,\infty)$ intervalda absolyut integrallanuvchi bo'lsin. Yuqoridagi (7) formulani quyidagicha yozamiz [2]:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt\right]e^{i\lambda x}d\lambda.$$

Ushbu funksiyadan belgili

$$F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt,$$

bunda

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\lambda)e^{i\lambda x}d\lambda.$$

$F(\lambda)$ funksiya $f(x)$ funksiyaning Fure akslantirishi deyiladi , $f(x)$ esa $F(\lambda)$ funksiyaning teskari Fure akslantirishi deyiladi.

Fure akslantirishi yordamida bir jinsli issiqlik o'tkazuvchanlik tenglamasi uchun Koshi masalasini yechishni ko'rsatamiz [3].

$$u_t = a^2 u_{xx}, \qquad -\infty < x < \infty, \quad t > 0 \qquad (8)$$

$$u(x,0) = \varphi(x), \qquad (9)$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt, \qquad \omega \in (-\infty, +\infty) \qquad (10)$$

(8)-tenglamaning ong tarafi x-o'zgaruvchi bo'yicha Fure akslantirishi quyidagicha bo'ladi:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U_t e^{-i\omega x} dx = \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U(x,t) e^{-i\omega x} dx\right] = V_t(\omega,t),$$

Bunda, $V(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U(x,t) e^{i\omega x} dx$ -izlanuvchi $U(x,t)$ funksiyaning obrazi. $x \to \pm\infty$ da barcha $t > 0$ ushun $U(x,t) \to 0, U_x(x,t) \to 0$ va $U(x,t)$ funksiya butun son o'qida absolyut integrallanadi. Unda (10)- Fure akslantirishi bo'yicha (8)-tenglamani x-o'zgaruvchi bo'yicha ikki marta bo'laklab integrallash natijasida quyidagiga ega bo'lamiz:

$$\frac{a^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U_{xx} e^{-i\omega x} dx = \frac{a^2}{\sqrt{2\pi}}\left[U_x e^{-i\omega x}\Big|_{x=-\infty}^{x=+\infty} + \underbrace{i\omega U e^{-i\omega x}\Big|_{x=-\infty}^{x=+\infty}}_{0} - \omega^2 \int_{-\infty}^{+\infty} U e^{-i\omega x} dx\right] =$$

$$= -a^2\omega^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U e^{-i\omega x} dx = -a^2\omega^2 V(\omega,t),$$

(9) boshlang'ich shartning Fure akslantirishi quyidagisha:

$V(\omega,0) = \Phi(\omega)$.

Bunda $\Phi(\omega) - \varphi(x)$ funksiyaning Fure obrazi,

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\xi) e^{-i\omega\xi} d\xi.$$

Demak, Fure akslantirishi yordamida (8)-(9)-Koshi masalasi quyidagi masalaga keladi:

$$\begin{cases} V_t + a^2\omega^2 V = 0, \\ V(\omega,0) = \Phi(\omega). \end{cases}$$

Uning yechimi $V(\omega,t) = \Phi(\omega)e^{-a^2\omega^2 t}$ izlanuvchi $U(x,t)$ funksiyaning Fure obrazı boladi. $U(x,t)$ funksiyaga qaytish uchun $V(\omega,t)$ ga teskari Fure akslantirishini qollash kerak bo'ladi,

$$U(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} V(\omega,t)e^{i\omega x}d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi(\omega)e^{-a^2\omega^2 t + i\omega x}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\xi)\int_{-\infty}^{+\infty} e^{-a^2\omega^2 t + i\omega(x-\xi)}d\omega d\xi =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\xi)\left[\int_{0}^{+\infty} e^{-a^2\omega^2 t + i\omega(x-\xi)}d\omega + \int_{-\infty}^{0} e^{-a^2\omega^2 t + i\omega(x-\xi)}d\omega\right]d\xi =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\xi)\int_{0}^{+\infty} e^{-a^2\omega^2 t}\left[\underbrace{e^{i\omega(x-\xi)} + e^{-i\omega(x-\xi)}}_{2\cos(x-\xi)}\right]d\omega d\xi = \frac{1}{\pi}\int_{-\infty}^{+\infty} \varphi(\xi)\int_{0}^{+\infty} e^{-a^2\omega^2 t}\cos(x-\xi)d\omega d\xi,$$

demak,

$$U(x,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \varphi(\xi)\int_{0}^{+\infty} e^{-a^2\omega^2 t}\cos(x-\xi)d\omega d\xi, \qquad (11)$$

integralni hisoblaymiz.

$$I(z) = \int_{0}^{+\infty} e^{-\alpha\omega^2}\cos\omega z d\omega, \qquad z > 0, \qquad (12)$$

$I'(z)$ hosilani hisoblab 1 marta bo'laklab integrallaymiz:

$$I'(z) = -\int_{0}^{+\infty} e^{-\alpha\omega^2}\omega\sin\omega z d\omega = \frac{1}{2\alpha}e^{-\alpha\omega^2}\omega\sin\omega z d\omega\Big|_{\omega=0}^{\omega=+\infty} - \frac{z}{2\alpha}\int_{0}^{+\infty} e^{-\alpha\omega^2}\cos\omega z d\omega =$$

$$= -\frac{z}{2\alpha}\int_{0}^{+\infty} e^{-\alpha\omega^2}\cos\omega z d\omega = -\frac{z}{2\alpha}I(z),$$

Bundan oddiy differensial tenglamaga kelamiz:

$$I'(z) + \frac{z}{2\alpha}I(z) = 0.$$

Uning yechimi

$$I(z) = Ce^{-\frac{z^2}{4\alpha}},$$

boladi. $C-$ni aniqlash ushun $I(0)-$dan foydalanamiz , Puasson integralidan,

$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}dy = 1$ - Puasson integrali.

$$I(0) = \int_{0}^{+\infty} e^{-\alpha\omega^2}d\omega = \frac{1}{\sqrt{\alpha}}\int_{0}^{+\infty} e^{-\lambda^2}d\lambda = \frac{1}{2\sqrt{\alpha}}\int_{-\infty}^{+\infty} e^{-\lambda^2}d\lambda = \frac{\sqrt{\pi}}{2\sqrt{\alpha}},$$

bundan $C = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}$ va

$$I(z) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}e^{-\frac{z^2}{4\alpha}}, \qquad (13)$$

(12)-da $\alpha = a^2t; z = x - \xi$ va (11) ni hisobga olsak, quyidagi yechimga ega bo'lamiz:

$$U(x,t) = \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}}e^{-\frac{(\xi-x)^2}{4a^2t}}\varphi(\xi)d\xi = \int_{-\infty}^{+\infty} C(x,\xi,t)\varphi(\xi)d\xi,$$

bunda,

$$C(x,\xi,t) = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{(\xi-x)^2}{4a^2t}}.$$

Foydalanilgan adabiyotlar

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