Radon akslantirishi yordamida matematik fizika masalalarini yechish

Shamuratov Damir

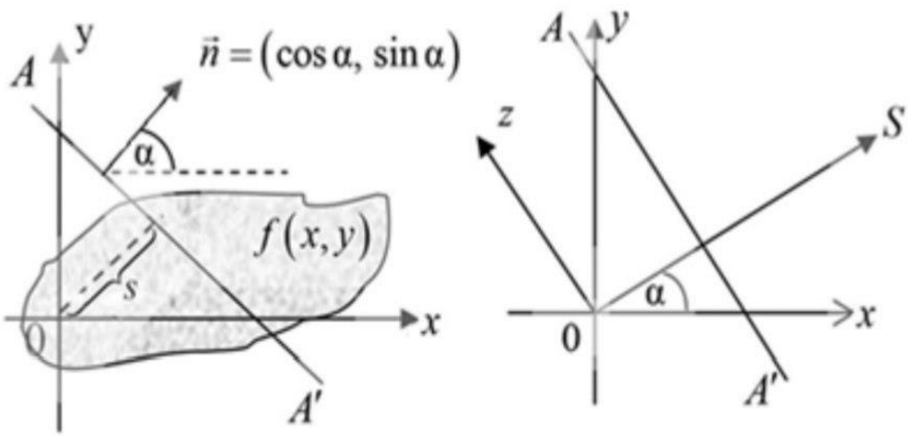
Berdaq nomidagi Qoraqalpoq Davlat Universiteti

1917-yil matematik I.Radon Fure akslantirishi bilan bog'liq bo'lgan ko'p o'zgaruvchili funksiyaning integral akslantirishini keltirib chiqardi. Radon akslantirishining asosiy xossalaridan biri – teskari akslanish , yaniy Radon akslantirishi orqali orginal funksiyaning qayta o'z holiga kelishi.

Ikki o'zgaruvchili funksiyani ko'rib chiqamiz. $f(x,y)$ butun tegislikda aniqlangan va cheksizlikda yetarlicha tez kamayib boruvchi ikkita haqiyqiy o'zgaruvchili funksiya bo'lsin. $f(x,y)$ funksiyaning Radon akslantirishi deb

$$R(s,\alpha)=\int_{L=AA'} f(x,y)dL=\int_{-\infty}^{\infty} f(s\cos\alpha - z\sin\alpha, s\sin\alpha + z\cos\alpha)dz, \qquad (1)$$

funksiyaga aytiladi [1]. Radon akslantirishining oddiy geometrik qiymati shundan iborat $\vec{n}=(\cos\alpha,\sin\alpha)$ vektoriga perpendikulyar bo'lgan AA' tog'ri chiziq bo'ylab $f(x,y)$ funksiyaning integrali bo'lib, koordinata boshidan s oraliqda o'tadi (1-rasm). AA' tog'ri chiziq tenglamasi : $x\cos\alpha + y\sin\alpha - s = 0$



1-rasm.

Yangi (s,z) o'zgaruvchilar eski (x,y) o'zgaruvchilardan soat tiliga qarama-qarshi α burchak bo'ylab aylantirish orqali olinadi. Analitik geometriyadan biz bilamiz $x = s\cos\alpha - z\sin\alpha, y = s\sin\alpha + z\cos\alpha$ bo'ladi, teskari akslantirish bo'lsa $s = x\cos\alpha + y\sin\alpha, z = -x\sin\alpha + y\cos\alpha$ ga teng.

Radon akslantirishi va Fure akslantirishi [2] orasidagi bog'liqlikni ko'rib chiqamiz. $f(x, y)$ funksiyaning ikki o'lchovli Fure akslantirishi quyidagicha

$$F(k_x, k_y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{-i(k_x x + k_y y)} f(x, y) dy. \qquad (2)$$

(x, y) o'zgaruvchidan (s, z) o'zgaruvchilarga o'tamiz va $\vec{k} = (k_x, k_y) = \omega(\cos\alpha, \sin\alpha)$ deb belgilaymiz, yaniy tegislikda $k_x O k_y$ polyar koordinatalar sistemasiga o'tamiz. (2) integraldagi o'zgaruvchilarni almashtirish orqali

$$F(\omega\sin\alpha, \omega\cos\alpha) = \int_{-\infty}^{\infty} ds e^{-i\omega s} \left(\int_{-\infty}^{\infty} f(s\cos\alpha - z\sin\alpha, s\sin\alpha + z\cos\beta) dz \right),$$

(3)

ga ega bo'lamiz. (3) da $k_x x + k_y y = \omega(x\cos\alpha + y\sin\alpha) = \omega s$ va Yakobiyani birga teng.

(1) ni hisobga olsak

$$F(\omega\cos\alpha, \omega\sin\alpha) = \int_{-\infty}^{\infty} e^{-i\omega s} R(s, \alpha) ds, \qquad (4)$$

ga teng. Ikki o'lchovli teskari Fure akslantirishidagi

$$f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} F(k_x k_y) dk_y \qquad (5)$$

o'zgaruvchilarni polyar koordinatalarida qayta yozamiz, yaniy

$$f(x, y) = \frac{1}{(2\pi)^2} \int_0^{\infty} \omega d\omega \int_0^{2\pi} e^{i\omega(x\cos\alpha + y\sin\alpha)} F(\omega\cos\alpha, \omega\sin\alpha) d\alpha. \qquad (6)$$

(4) ni hisobga olsak (6) bizga teskari Radon akslantirish formulasini beradi:

$$f(x, y) = \frac{1}{(2\pi)^2} \int_0^{\infty} \omega d\omega \int_0^{2\pi} e^{i\omega(x\cos\alpha + y\sin\alpha)} \tilde{R}(\omega, \alpha) d\alpha, \qquad (7)$$

bu yerda

$$\tilde{R}(\omega,\alpha) \equiv F(\omega\cos\alpha, \omega\sin\alpha) = \int_{-\infty}^{\infty} R(s,\alpha)e^{i\omega s}ds. \quad (8)$$

R^2 – to'lqin tenglamasi uchun Koshi masalasini yeching [3]:

$$\begin{cases} U_{tt} = a^2\Delta U(x), & (9) \\ U(x,0) = 0, & (10) \\ U_t(x,0) = f(x), & (11) \end{cases}$$

$$x \in R^2; \; x = (x_1, x_2); \; \Delta = \frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}}.$$

Bu masala Radon akslantirishi yordamida yechiladi, yaniy bu Koshi masalasining yechimini Radon akslantirishi bo'yicha funksiyani tenglash orqali topishga bo'ladi.

Bir o'lchamli holatdagidek ikkita uzluksiz hosilaga ega $f = f(t)$ bir o'zgaruvchili funksiyani ko'ramiz. Unda yechim "yassi to'lqin " ko'rinishida bo'ladi.

$U(x,t) = f\left(\langle x,\xi\rangle + t\right)$, $\langle x,\xi\rangle$ – skalyar ko'paytma, $|\xi| = 1.$

Demak, $U(x,t)$ (9)-tenglamani qanoatlandiradi, buni tekshiramiz:

$$U_t = \pm af'\left(\langle x,\xi\rangle + at\right),$$

$$U_{tt} = (\pm a)^2 f''\left(\langle x,\xi\rangle \pm at\right) = a^2 f''\left(\langle x,\xi\rangle \pm at\right). \quad (12)$$

$$U_{x_1} = \xi_1 f'\left(\langle x,\xi\rangle \pm at\right),$$

$$U_{x_1x_1} = \xi_1^{\,2} f''\left(\langle x,\xi\rangle \pm at\right),$$

$$U_{x_2x_2} = \xi_2^{\,2} f''\left(\langle x,\xi\rangle \pm at\right).$$

$$\Delta V = f''\left(\langle x,\xi\rangle \pm at\right)\cdot\left(\xi_1^{\,2} + \xi_2^{\,2}\right) = f''\left(\langle x,\xi\rangle \pm at\right), \quad (13)$$

(9)-tenglamaga olib borib qoysak (12) va (13) lardan,

$$a^2 f''\left(\langle x,\xi\rangle \pm at\right) = a^2 f''\left(\langle x,\xi\rangle \pm at\right),$$

bo'ladi. Demak, ikki marta uzluksiz differensiallanuvchi xohlagan funksiya , agar uning argumenti $\langle x,\xi\rangle \pm at$ ko'rinishida bo'lsa (9)-tenglamaning yechimi bo'ladi. Agar

$U(x,0) = f(x),$

boshlang'ich berilgan bo'lsa, unda to'lqin tenglamasining yechimi shu shartlarni qanoatlandiradi va quyidagicha bo'ladi :

$$U(x,t) = \frac{f\left(\langle x,\xi\rangle + at\right) + f\left(\langle x,\xi\rangle - at\right)}{2}.$$

Radon akslantirishi $p-$o'zgaruvchi bo'yicha "yassi to'lqin" ko'rinishidagi funksiyani beradi,

$$R\left[T^{y} f(x)\right]\cdot(\xi;p) = R[f]\left(\xi;p+\langle\xi,y\rangle\right).$$

Quyidagi Radon akslantirishining funksiyasini ko'rib o'tamiz:

$$U(x,t) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{dp}{p}\int_0^{2\pi} R[f]\left(\xi, p-t+\langle x,\xi\rangle\right)d\varphi, \qquad (14)$$

bunda, $\xi-$vektor, $\varphi-$burchak koordinata funksiyasi bo'ladi

$\xi = \xi(\varphi) = (\cos\varphi, \sin\varphi).$

Demak, (14)-funksiya (9)-(11) Koshi maslasining yechimi bo'ladi. Bu funksiya (9)-tenglamani qanoatlandiradi, sababi (14)-"yassi to'lqin" tipidagi funksiya. Endi (10)-shartni tekshiramiz:

$$U(x,0) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_0^{2\pi}\frac{1}{p}(R,f)\left(\xi, p+\langle x,\xi\rangle\right)d\varphi dp =$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_0^{\pi}\frac{1}{p}(R,f)\left(\xi, p+\langle x,\xi\rangle\right)d\varphi dp +$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_0^{\pi}\frac{1}{p}(R,f)\left(-\xi, p-\langle x,\xi\rangle\right)d\varphi dp.$$

Keyingi integralda $p-$o'zgaruvchini $(-p)-$ga o'zgartiramiz, unda:

$$U(x,0) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_0^{\pi}\frac{1}{p}(R,f)\left(\xi, p+\langle x,\xi\rangle\right)d\varphi dp +$$

$$+\frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty}\int\limits_0^{\pi}\frac{1}{(-p)}\left(R,f\right)\left(-\xi,-p+\left\langle x,\xi\right\rangle\right)d\varphi d(-p)=$$

$$=\frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty}\int\limits_0^{\pi}\frac{1}{p}\left(R,f\right)\left(\xi,p+\left\langle x,\xi\right\rangle\right)d\varphi dp-$$

$$-\frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty}\int\limits_0^{\pi}\frac{1}{p}\left(R,f\right)\left(-\xi,-p-\left\langle x,\xi\right\rangle\right)d\varphi d(-p).$$

$\left\langle x,\xi\right\rangle=p \Rightarrow -\left\langle x,\xi\right\rangle=-p$ ga teng ku'chli bo'ladi, yaniy Radon akslantirishida p ni $(-p)$ ga, ξ ni $(-\xi)$ ga almashtirganda ifodaning qiymati o'zgarmaydi. Unda (10)-shartni qanoatlandiradi, yaniy

$U(x,0)=0,$

bo'ladi. Endi (11)-shartni tekshiramiz:

$$U_t(x,t)\Big|_{t=0}=-\frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty}\frac{1}{p}\left(R,f\right)_p'\left(p+\left\langle x,\xi\right\rangle\right)d\varphi d\rho.$$

Bu Radon akslantirishining o'rta sferik akslantirish formulasi bo'ladi, bundan

$U_t(x,0)=f(x).$

Demak, (14)-funksiya (9)-(11) Koshi masalasining yechimi bo'ladi.

Foydalanilgan adabiyotlar

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